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Frank Cowell: Microeconomics
Solution to Exercise 2.6

The exercise discusses the CES (constant elasticity of substitution) production function. The solution contains a couple of erroneous, or at least misleading, statements.

In the exercise, the production function is given by

(1) $$\varphi(\mathbf{z}) = \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}},$$

where z_i is the quantity of input *i* and $\alpha_i \geq 0$, $-\infty < \beta \leq 1$ are parameters.

The right-hand side of (1) is not defined for $\beta = 0$. The expression makes sense for $\beta \in (-\infty, 0) \cup (0, 1]$.

Below, $\alpha_i > 0$ and $z_i > 0$ will be assumed. If $\alpha_i = 0$ is permitted, this case must be singled out and discussed separately in some of the arguments below, complicating the exposition without contributing to improved understanding. The same holds for $z_i = 0$. The possibility $z_i < 0$ can be ruled out by general assumptions made in the textbook.

The elasticity of substitution is computed and found to be $\sigma = \dfrac{1}{1-\beta}$, from which follows $\lim\limits_{\beta \to -\infty} \sigma = 0$, $\lim\limits_{\beta \to 0} \sigma = 1$ and $\lim\limits_{\beta \to 1} \sigma = \infty$. This justifies the claim that these limiting cases correspond to the Leontief, the Cobb-Douglas and the linear production functions, respectively.

It appears, however, that the solution makes stronger statements, by specifying the value of $\lim \varphi(\mathbf{z})$ in each of the three cases. The following seems to be claimed:

(2) $$\lim_{\beta \to -\infty} \varphi(\mathbf{z}) = \min\{\alpha_1 z_1, \alpha_2 z_2\}$$

(3) $$\lim_{\beta \to 0} \varphi(\mathbf{z}) = z_1^{\alpha_1} z_2^{\alpha_2}$$

(4) $$\lim_{\beta \to 1} \varphi(\mathbf{z}) = \alpha_1 z_1 + \alpha_2 z_2$$

Of these statements, only (4) is true for all admissible values of α_i. The cases (2) and (3) are discussed below.

<u>Leontief</u>

Here $\beta < 0$ can be assumed. Moreover, let $z_1 \leq z_2$. Expression (1) can be rewritten

(5) $$\varphi(\mathbf{z}) = \left[z_1^\beta \left\{\alpha_1 + \alpha_2 \left(\frac{z_2}{z_1}\right)^\beta\right\}\right]^{\frac{1}{\beta}} = z_1 \cdot \left[\alpha_1 + \alpha_2 \left(\frac{z_2}{z_1}\right)^\beta\right]^{\frac{1}{\beta}}.$$

Under the stated assumptions, $0 < \left(\frac{z_2}{z_1}\right)^{\beta} \le 1$, implying $\alpha_1 < \alpha_1 + \alpha_2 \left(\frac{z_2}{z_1}\right)^{\beta} \le \alpha_1 + \alpha_2$.

Since $\frac{1}{\beta} < 0$, this gives

$$\text{(6)} \qquad \alpha_1^{\frac{1}{\beta}} > \left[\alpha_1 + \alpha_2 \left(\frac{z_2}{z_1}\right)^{\beta}\right]^{\frac{1}{\beta}} \ge \left[\alpha_1 + \alpha_2\right]^{\frac{1}{\beta}}$$

For all $K > 0$, $\lim_{\beta \to -\infty} K^{\frac{1}{\beta}} = 1$. In particular, $\lim_{\beta \to -\infty} \alpha_1^{\frac{1}{\beta}} = \lim_{\beta \to -\infty} \left[\alpha_1 + \alpha_2\right]^{\frac{1}{\beta}} = 1$. Then (5) and (6) imply $\lim_{\beta \to -\infty} \varphi(\mathbf{z}) = z_1$. Similarly, if $z_1 \ge z_2$. then $\lim_{\beta \to -\infty} \varphi(\mathbf{z}) = z_2$. The conclusion is

$$\lim_{\beta \to -\infty} \varphi(\mathbf{z}) = \min\{z_1, z_2\}.$$

In other words, (2) holds if and only if $\alpha_1 = \alpha_2 = 1$.

Cobb-Douglas

The right-hand side of (1) is homogeneous of degree 1 in **z**, regardless of the values of the parameters. The right-hand side of (3) is homogeneous of degree $\alpha_1 + \alpha_2$ in **z**. Hence (3) can only hold when $\alpha_1 + \alpha_2 = 1$.

Taking the natural logarithm of both sides in (1) gives

$$\text{(7)} \qquad \ln\{\varphi(\mathbf{z})\} = \frac{\ln\left[\alpha_1 z_1^{\beta} + \alpha_2 z_2^{\beta}\right]}{\beta}.$$

For all $K > 0$, $\lim_{\beta \to 0} K^{\beta} = 1$. Hence $\lim_{\beta \to 0}\left[\alpha_1 z_1^{\beta} + \alpha_2 z_2^{\beta}\right] = \alpha_1 + \alpha_2$. Now assume $\alpha_1 + \alpha_2 = 1$. Then the right-hand side of (7) is a $\frac{0}{0}$-expression as β tends to 0, and $\lim_{\beta \to 0}\left[\ln\{\varphi(\mathbf{z})\}\right]$ can be found by l'Hôpital's rule, that is, by differentiating the numerator and denominator of the right-hand side of (7) with respect to β. The derivative of the numerator in (7) is

$$\frac{\left(\alpha_1 \cdot \ln z_1 \cdot z_1^{\beta} + \alpha_2 \cdot \ln z_2 \cdot z_2^{\beta}\right)}{\alpha_1 z_1^{\beta} + \alpha_2 z_2^{\beta}},$$

which tends to $\alpha_1 \cdot \ln z_1 + \alpha_2 \cdot \ln z_2$ as β tends to 0. The derivative of the denominator in (7) is 1. Hence $\lim_{\beta \to 0}\left[\ln\{\varphi(\mathbf{z})\}\right] = \alpha_1 \cdot \ln z_1 + \alpha_2 \cdot \ln z_2$, from which (3) follows.

That is, (3) holds if and only if $\alpha_1 + \alpha_2 = 1$.

The following statements, which contradict (3), can be deduced from (7):

$$\alpha_1 + \alpha_2 < 1 \Rightarrow \lim_{\beta \to 0}\left[\ln\{\varphi(\mathbf{z})\}\right] = -\infty \Rightarrow \lim_{\beta \to 0} \varphi(\mathbf{z}) = 0$$

$$\alpha_1 + \alpha_2 > 1 \Rightarrow \lim_{\beta \to 0}\left[\ln\{\varphi(\mathbf{z})\}\right] = \infty \Rightarrow \lim_{\beta \to 0} \varphi(\mathbf{z}) = \infty$$